THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Suggested Solutions to Homework 4

4. Show that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if every subsequence of (x_n) has in turn a (sub)subsequence that converges to x.

   Proof. " $\Longrightarrow$ " Suppose $x_n \to x \in \mathbb{R}$. Let (x_{n_k}) be any subsequence of (x_n). Then x_{n_k} converges to x, which is a subsequence of itself that converges to x.

   " $\Longleftarrow$ " We will prove by contradiction. Suppose (x_n) does not converge to x. Then there exists $\epsilon_0 > 0$ and a subsequence (x_{n_k}) which satisfies $|x_{n_k} - x| \geq \epsilon_0$ for any $k \in \mathbb{N}$. But now by assumption, (x_{n_k}) has a subsequence $(x_{n_{k_j}})$ which converges to x. Take $\epsilon := \epsilon_0 > 0$, then there exists $J \in \mathbb{N}$ such that for any $j \geq J$, we have $|x_{n_{k_j}} - x| < \epsilon_0$. This is a contradiction to our choice of (x_{n_k}), and hence (x_n) must converge to x. $\square$

5. Let (x_n) be a bounded sequence that does not converge to $x \in \mathbb{R}$. Show that there exists a subsequence of (x_n) that converges to some $x' \neq x$.

   Proof. Since (x_n) does not converge to x, there exists $\epsilon_0 > 0$ and a subsequence (x_{n_k}) which satisfies $|x_{n_k} - x| \geq \epsilon_0$ for any $k \in \mathbb{N}$. Since (x_n) is bounded, (x_{n_k}) is also bounded. By Bolzano-Weierstrass Theorem, there is a subsequence $(x_{n_{k_j}})$ of (x_{n_k}) that converges to some $x' \in \mathbb{R}$. What remains to show is that $x' \neq x$. But if x' were equal to x, take $\epsilon := \epsilon_0 > 0$, then there exists $J \in \mathbb{N}$ such that for any $j \geq J$, we have $|x_{n_{k_j}} - x| < \epsilon_0$. This is a contradiction to our choice of (x_{n_k}). Hence $x' \neq x$. Therefore the subsequence $x_{n_{k_j}}$ converges to $x' \neq x$. $\square$